Journal of Statistical Physics, Vol. 98, Nos. 1/2, 2000

Decoherent Histories and Realism

Angelo Bassi[1] **and GianCarlo Ghirardi**[2]

Received August 7, 1999; final August 13, 1999

We reconsider the decoherent histories approach to quantum mechanics and analyze some problems related to its interpretation which we believe have not been adequately clarified by its proponents. We put forward some assumptions which, in our opinion, are necessary for a realistic interpretation of the probabilities that the formalism attaches to decoherent histories. We prove that such assumptions, unless one limits the set of the decoherent families which can be taken into account, lead to a logical contradiction. The line of reasoning we follow is conceptually different from other arguments which have been presented and which have been rejected by the supporters of the decoherent histories approach. The conclusion is that the decoherent histories approach, to be considered as an interesting realistic alternative to the orthodox interpretation of quantum mechanics, requires the identification of a mathematically precise criterion to characterize an appropriate set of decoherent families which does not give rise to any problem.

KEY WORDS: Interpretation of quantum mechanics; decoherent histories; probabilities and truth values.

1. INTRODUCTION

After more than 70 years of debate about the difficulties that one encounters in working out a coherent view of physical processes based on the standard formulation of quantum mechanics, there is now a widespread belief that such difficulties do not arise from philosophical prejudices (as has been repeatedly asserted by many of the supporters of *textbook quantum mechanics*) but represent precise mathematical and physical challenges which call for a physical solution. As J. Bell appropriately stated[(1)] *the way*

[1] Department of Theoretical Physics of the University of Trieste, Trieste, Italy; e-mail: bassi@ts.infn.it.

[2] Department of Theoretical Physics of the University of Trieste, and the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy; e-mail: ghirardi@ts.infn.it.

ahead is unromantic in that it requires mathematical work by theoretical physicists, rather than interpretations by philosophers. It is also encouraging, for those who share this position, to see that now there are explicit proposals indicating possible ways out in which the process of measurement (and more generally all those measurement-like processes[(2)] *we are obliged to admit ... are going on more or less all the time*, *more or less everywhere*) is analyzed not in terms of vague assertions or resorting to ill defined and/or contradictory dualistic evolution processes (the linear and deterministic evolution for microsystems and the nonlinear and stochastic wave packet reduction for the macroscopic ones) but in terms of more fundamental physical concepts. In particular, approaches of this kind, which S. Goldstein[(3)] has appropriately denoted as *observer-free formulations of quantum mechanics*, can be grouped in three categories: the Pilot-Wave Theories, the Decoherent Histories and the Spontaneous Localization models.

This paper is devoted to a critical analysis of the so called Decoherent Histories (DH) approach.[(4–14)] The motivation of such an approach is one we completely share, i.e., to transform the probabilistic statements of quantum theory which, in its standard formulation, are not referring to properties actually possessed by individual physical systems but to the potentialities of getting certain outcomes conditional under the measurement being performed, into statements about sequences (histories) of objective events. To each history the DH approach attaches a probability which coincides with the one that standard quantum mechanics attaches to a process in which the measurements aimed to ascertain the claimed properties are actually performed and give the stated results.

The DH approach faces various problems, the most crucial deriving from quantum interference effects, which render problematic the coarse-graining of histories. The way out is obtained by assigning probabilities not to all conceivable histories, but only to an appropriately selected subset of them. Actually the procedure consists in adopting a precise mathematical criterion, the decoherence condition, which allows the identification of families of *decoherent histories*, closed under coarse-graining. Only to histories belonging to such families the formalism attaches definite probabilities.

Even when this move is done, the proposed DH interpretation of the probabilities meets two serious difficulties. The first one originates from the fact that the decoherence condition, in spite of the fact that it limits significantly the set of the acceptable families, turns out not to be sufficiently restrictive. For instance, Dowker and Kent[(15)] have proved that within the theory, *taking the past and present for granted*, one can identify infinitely many decoherent families of histories which imply, in general, a future nonclassical behaviour for macroscopic systems: that is, the theory has a very weak predictive power. Quite in general, the very existence of *physically senseless decoherent families*[3] raises serious problems of interpretation. Which meaning whatsoever can one attach to the histories of a decoherent family claiming that the celebrated Schrödinger's cat is not **either** *alive* **or** *dead, but alive + dead*?

The second problem is related to the occurrence of incompatible decoherent families.[4] This fact raises serious problems of interpretation whose analysis represents a relevant part of the present paper. For the moment we confine ourselves to recall that, according to Griffiths,[(4–8)] the correct way to circumvent the difficulties consists in stating that all reasonings, all conclusions about properties possessed by physical systems, hold only when they are drawn within a **single family** of decoherent histories, or, at most, within compatible families (we will denote this prescription as "the single family rule"). Otherwise, any assertion is devoid of any meaning whatsoever. This fact is rather puzzling and gives rise to problems with classical logic, as stressed, e.g., by d'Espagnat.[(16)] Actually, Griffiths[(7, 8)] and Omnès[(10, 11)] themselves have felt the necessity, to face this problem, to make more precise the logical and interpretative bases of the theory.

The present paper consists of two parts and is organised in the following way: the first part is devoted to present a general sketchy view of the DH approach aimed to focus its most relevant features which will be the subject of the subsequent discussion. In Section 2 we recall the formal aspects of the theory with particular reference to the crucial role played by the decoherence condition and to its physical meaning. We will also resort to an elementary example to better clarify this point. In Section 3, following the line of thought of Griffiths[(7)] and Omnès,[(10)] we tackle the problem of making more explicit the logical structure of the theory by equipping the histories of a decoherent family with the structure of a Boolean algebra. This step will clarify the sense in which the scheme should allow to recover classical logic and reasoning. Particular attention will be devoted (taking into account the fundamentally probabilistic structure of the theory) to the formalization of the idea of "logical implication" between histories, a crucial step to work out a sensible "quantum reasoning." We will also show,

[3] I.e., families whose histories are manifestly unacceptable on physical grounds.

[4] Two decoherent families are said to be incompatible when one cannot combine them in a larger decoherent family. According to the very spirit of the DH approach, inconsistencies deriving from comparing different histories belonging to incompatible families are irrelevant: just as one cannot compare statements referring to noncommuting observables in standard quantum mechanics, one cannot derive conclusions from arguments requiring the consideration of different histories belonging to incompatible families.

by resorting once more to an elementary example, how, to avoid inconsistencies, one has to accept "the single family rule." In the final section of this first part of the paper we raise the problem of how many decoherent families have to be taken into account and we outline the different positions of different supporters of the DH approach about this problem.

In Part II of the paper we deepen our analysis by raising the fundamental (in our opinion) question of the nature of the probabilities of the DH approach. In particular, in Section 4 we discuss whether one can attach truth values to the probabilistic statements of the theory. By resorting to elementary examples taken from Classical and Standard Quantum mechanics we try to make clear that only a precise position about this crucial problem allows to really overcome the conceptual difficulties of the orthodox interpretation of quantum mechanics. In fact, the question of whether the histories of a decoherent family have truth values has a direct relation to the question of whether the statements of the theory refer to properties of individual physical systems which can be considered as objectively possessed by them. The mathematically natural way to implement the idea of the histories possessing truth values is that of assuming that there exist an appropriate homomorphism between the histories and the set $\{0, 1\}$, the two values corresponding to the falsity and the truth of a given history, i.e., of the statements it makes about properties possessed (in general at different times) by an individual physical system. By resorting once more to elementary physical examples, we stress, first of all that truth values can be attached only to histories belonging to at least one decoherent family. Thus, we will consider a precise homomorphic map of the members of any decoherent family on the set $\{0, 1\}$, the map being, *a priori*, family dependent.

This approach will allow us to discuss the relations existing between the homomorphism we are interested in and the "single family rule" analyzed in Part I of the paper. Since the proponents of the DH approach have not committed themselves about the truth values of the histories, no immediate connection can be established between such a rule and the basic features of the homomorphism we are interested in (contrary to the opinion of Griffiths—private correspondence—and also of the referee of this paper). Accordingly, we will investigate critically the problem of truth assignments to the histories of any given decoherent family.

At this stage a new problem arises: the *same history* (whose probability is family independent) may belong to different and incompatible decoherent families.[5] This leads naturally to raise the question of whether

[5] Actually, given any history belonging to at least one decoherent family there are always other incompatible decoherent families containing it.

the homomorphisms (one for each decoherent family) we have been led to consider in order to give a "classical" status to the probabilistic assertions of the theory might assign different truth values (depending on the decoherent family to which it is considered to belong) to the same history. We argue that this cannot be the case unless one is keen to spoil the theory of all its most appealing features. Accordingly, we put forward the assumption that the truth value which is attached to a specific history belonging to a decoherent family is independent from the particular family one has in mind. We stress that nothing in the original formulation of the "single family rule" implies that such an assumption is illegitimate for the simple reason that the proponents of the DH approach did not face the problem of the truth values of the histories. However, if somebody feels that this requirement violates the single family rule (we stress that we have deliberately chosen to use the term "feels" because in the formulation of such a rule nothing implies—syntactically or semantically—the exclusion of such an assumption) then we are ready to plainly declare that we are violating this particular reading of the rule. But we will make clear that the theory, if one does not equip it with an homomorphism making true or false its statements or if one accepts that the same history can have different truth values, does not exhibit any conceptual advantage with respect to the standard interpretation of quantum mechanics. Then, we prove a general theorem, i.e., that the assumptions:

(a) Every decoherent family can be equipped with a natural Boolean structure reflecting the Boolean structure of classical logic,

(b) Every history belonging to a decoherent family has a precise truth value,

(c) The truth value of any precise single history does not depend on the decoherent family to which it belongs,

(d) All decoherent families have the same status and must be taken into account,

lead to a contradiction.

The last section is devoted to a brief review of the main interpretative problems analyzed in this paper. In our opinion the most appropriate choice consists in limiting the class of all decoherent families which can be taken into account; a similar conclusion was reached by the authors of refs. 17 and 3. We call attention to the fact that even this program is not easy to implement, but we leave open the question of whether one can reach this goal in a satisfactory way. We will reconsider the problem of further limiting the acceptable families in a subsequent paper.

PART I

2. THE DECOHERENT HISTORIES

In this section we present a concise summary of the Decoherent Histories approach, and we show, by resorting to a simple example, how one can argue about the properties and about the time evolution of quantum systems within such a scheme.

2.1. The Formalism

We consider an individual physical system S—a particle, a macroscopic object, the whole universe—whose initial state at time t_0 is described by the statistical operator $W(t_0) = W$. Let $\mathscr{H}$ be the Hilbert space of the system and $U(t, t')$ the unitary operator describing its evolution.

We also consider an ordered sequence of time instants, $(t_1, t_2,..., t_n)$, $t_m < t_{m+1}$, $(m = 1, 2,..., n-1)$ and, associated to any time t_m of the sequence, an exhaustive and exclusive set[6] of projection operators $\{P_k^{(m)}\}$:

$$\sum_k P_k^{(m)} = 1, \qquad P_k^{(m)} P_j^{(m)} = \delta_{k,j} P_k^{(m)} \tag{2.1}$$

Besides these operators we will take also into account all their possible sums:

$$Q_{\alpha_m}^{(m)} = \sum_k \pi_{\alpha_m}^k P_k^{(m)} \tag{2.2}$$

where $\pi_{\alpha_m}^k$ takes the values 0 or 1. Obviously, when all possible values of $\pi_{\alpha_m}^k$ are considered, one gets 2^N different projection operators $Q_{\alpha_m}^{(m)}$ for each time t_m, where N represents the number of the projection operators $P_k^{(m)}$ of the exhaustive and exclusive set (2.1).

Definition. A *History* $\text{His}^{(\alpha)}$ is a sequence of n pairs $(Q_{\alpha_m}^{(m)}, t_m)$, $(m = 1, 2,..., n)$ each consisting of one projection operator from the set $\{Q_{\alpha_m}^{(m)}\}$ and the corresponding time t_m:

$$\text{His}^{(\alpha)} = \{(Q_{\alpha_1}^{(1)}, t_1), (Q_{\alpha_2}^{(2)}, t_2),..., (Q_{\alpha_n}^{(n)}, t_n)\} \tag{2.3}$$

α being a shorthand for the definite sequence $(\alpha_1, \alpha_2,..., \alpha_n)$.

[6] It is important to keep in mind that the families of projection operators associated to different times are, in general, different from each other.

Obviously, the history $\text{His}^{(\alpha)}$ is assumed to correspond to the statement: the physical system S possesses, at time t_1, the properties described by the projection operator $Q^{(1)}_{\alpha_1}$, at time t_2 those described by $Q^{(2)}_{\alpha_2}$, and so on. When the projection operators appearing in Eq. (2.3) belong to the basic sets $\{P^{(m)}_k\}$, $(m = 1, 2,..., n)$, the associated $\text{His}^{(\alpha)}$ are usually referred to as *fine-grained histories*, otherwise they are called *coarse-grained histories.*

Definition. A *Family of Histories* Fam is the set of all histories of the form (2.3), when the projection operators appearing in it run over all possible members of the sets $\{Q^{(m)}_{\alpha_m}\}$, $m = 1, 2,..., n$.

To each conceivable history (i.e., to any element of a family) one associates a precise weight which is assumed (if we understand correctly the aims of the proponents of the DH approach) to represent, under certain precise assumptions we will state at short, *the probability that the considered physical system actually possesses, at the associated times, the properties identified by the projection operators appearing in it.* The natural candidate for such a probability is:

$$p[\text{His}^{(\alpha)}] = \text{Tr}\{Q^{(n)}_{\alpha_n} U(t_n, t_{n-1})\, Q^{(n-1)}_{\alpha_{n-1}} U(t_{n-1}, t_{n-2}) \cdots U(t_1, t_0)\, W \times U^\dagger(t_1, t_0) \cdots U^\dagger(t_{n-1}, t_{n-2})\, Q^{(n-1)}_{\alpha_{n-1}} U^\dagger(t_n, t_{n-1})\, Q^{(n)}_{\alpha_n}\} \tag{2.4}$$

Note that this choice amounts to attach to the considered history the probability that standard quantum mechanics assigns to the process in which the system under consideration is subjected, at the chosen times, to the specified measurements, and one always gets the outcome $+1$ for the associated projectors. In this way one ensures from the very beginning that the theory is predictively equivalent to standard quantum mechanics with the orthodox interpretation. In order that the proposed probabilistic interpretation be tenable, $p[\text{His}^{(\alpha)}]$ must satisfy the usual probability rules; this happens iff Griffiths' condition:[4]

$$\text{Re}[\text{Tr}\{P^{(n)}_k U(t_n, t_{n-1})\, P^{(n-1)}_j U(t_{n-1}, t_{n-2}) \cdots U(t_1, t_0)\, W \times U^\dagger(t_1, t_0) \cdots U^\dagger(t_{n-1}, t_{n-2})\, P^{(n-1)}_{\tilde{j}} U^\dagger(t_n, t_{n-1})\, P^{(n)}_{\tilde{k}}\}] = 0 \tag{2.5}$$

holds whenever at least one of the elements of the sequence $(k, j,...)$ differs from the corresponding element of the sequence $(\tilde{k}, \tilde{j},...)$. Condition (2.5) is known as the *consistency condition*[4] or the *weak decoherence condition.*[13] If it is satisfied, the corresponding family of histories Fam is said to be consistent (or weakly decoherent).

In ref. 12, Gell-Mann and Hartle have introduced a stronger consistency condition, sometimes called the *medium decoherence condition*:

$$
\begin{aligned}
D(k, j,..., \tilde{k}, \tilde{j},...) = \mathrm{Tr}\{ P_k^{(n)} U(t_n, t_{n-1}) P_j^{(n-1)} U(t_{n-1}, t_{n-2}) \cdots U(t_1, t_0) W \\
\times U^\dagger(t_1, t_0) \cdots U^\dagger(t_{n-1}, t_{n-2}) P_{\tilde{j}}^{(n-1)} U^\dagger(t_n, t_{n-1}) P_{\tilde{k}}^{(n)} \} \\
= \delta_{k,\tilde{k}} \delta_{j,\tilde{j}} \cdots D(k, j,...; k, j,...)
\end{aligned}
\tag{2.6}
$$

with obvious meaning of the symbols. The quantity $D(k, j,...; \tilde{k}, \tilde{j},...)$ itself is called the *decoherence functional*. We point out that while the medium decoherence condition requires that the off-diagonal elements of the decoherence functional vanish, the weak decoherence condition requires only that their real parts vanish.[7] From now on we will always use condition (2.6) and we will refer to it, for simplicity, as the *decoherence condition* without any further specification. Accordingly, we introduce the following definition: *a family of histories is said to be decoherent if its fine-grained histories satisfy condition* (2.6).

It is useful to remark that the decoherence condition is a quite strict requirement on Fam and that in almost all cases (even many of the physically interesting ones) a family of histories is never exactly decoherent. For this reason Gell-Mann and Hartle themselves[(12)] introduced the idea of approximate decoherence, amounting to require that:

$$
D(k, j,...; \tilde{k}, \tilde{j},...) \approx \delta_{k,\tilde{k}} \delta_{j,\tilde{j}} \cdots D(k, j,...; k, j,...) \tag{2.7}
$$

In this paper we will analyse only the case of exactly decoherent histories; the analysis of the case of approximate decoherence and its physical consequences will be the subject of a subsequent paper.

2.2. The Physical Meaning of the Decoherence Condition

The requirements embodied in condition (2.5) or (2.6), i.e., that quantum probabilities behave like classical probabilities, are necessary if one takes into account that the interpretation pretends to make probabilistic claims about properties which are exhaustive, mutually exclusive and which are claimed to be possessed by a physical system independently of any observer, or any measurement process. However, as natural as they can appear, such requests are not satisfied, in general, by the histories constituting a family. That this is the case can be easily understood by taking into account the peculiar probabilistic structure of standard quantum mechanics. One can exhibit a quite elementary example which makes this

[7] Actually, Gell-Mann and Hartle have considered[(13)] also a stronger decoherence condition.

point perfectly clear by making reference to the famous two-slits experiment, *the only mystery of quantum mechanics*, in Feynman's words.[18]

Suppose one has a particle in an initial state corresponding to a wave function which is appreciably different from zero and almost constant on a large interval in the x-direction orthogonal to the direction along which the particle propagates and which impinges on a screen with two slits δ_1 and δ_2 of equal extension around the points x_1 and x_2. Let us denote by C the complement in the real axis x of the set $\delta_1 \cup \delta_2$. We also consider an infinite sequence $\{\Delta_j\}$ of disjoint intervals covering the whole x-axis. Let us denote by P_{δ_1}, P_{δ_2}, P_C and P_{Δ_j} the operators projecting on the closed linear manifolds of the square-integrable functions of x with support entirely contained in the indicated intervals. Finally we denote as $P_{\delta_1 \cup \delta_2}$ the coarse-grained projection operator on the manifold associated to the set $\delta_1 \cup \delta_2$. Let us now consider the family $\text{F}_{\text{AM}}^{(2\text{slits, 2times})}$ of two-times histories characterised by $\{P_k^{(1)}\} \equiv (P_{\delta_1}, P_{\delta_2}, P_C)$ and $\{P_j^{(2)}\} \equiv \{P_{\Delta_j}\}$. Such a family contains the coarse-grained history:

$$\text{H}_{\text{IS}}^{([\delta_1 \cup \delta_2] \& \Delta_j)} = \{(P_{\delta_1 \cup \delta_2}, t_1), (P_{\Delta_j}, t_2)\} \tag{2.8}$$

as well as the fine grained histories:

$$\text{H}_{\text{IS}}^{(\delta_1 \& \Delta_j)} = \{(P_{\delta_1}, t_1), (P_{\Delta_j}, t_2)\}, \quad \text{H}_{\text{IS}}^{(\delta_2 \& \Delta_j)} = \{(P_{\delta_2}, t_1), (P_{\Delta_j}, t_2)\} \tag{2.9}$$

It is then obvious that, due to quantum interference:

$$p(\text{H}_{\text{IS}}^{([\delta_1 \cup \delta_2] \& \Delta_j)}) \neq p(\text{H}_{\text{IS}}^{(\delta_1 \& \Delta_j)}) + p(\text{H}_{\text{IS}}^{(\delta_2 \& \Delta_j)}) \tag{2.10}$$

i.e., the considered family $\text{F}_{\text{AM}}^{(2\text{slits, 2times})}$ is not decoherent.

Note that the relation (2.10), implying that the decoherence condition is not satisfied, expresses the well known fact that the probability of a particle being within a certain interval at time t_2 subsequent to its passage through the screen with two-slits, does not coincide with the sum of the probability of being within the considered interval having been within the slit δ_1 at time t_1, plus the probability of being within the considered interval having been within the slit δ_2 at the same time.

Before concluding, we remark that, contrary to what happens in the case just considered, the families: $\text{F}_{\text{AM}}^{(2\text{slits}, t_1)}$ consisting of one-time histories at time t_1 whose fine grained elements are identified by the set $\{P_k^{(1)}\} \equiv \{P_{\delta_1}, P_{\delta_2}, P_C\}$, as well as the family $\text{F}_{\text{AM}}^{(2\text{slits}; t_1, t_2)}$ consisting of histories at times t_1 and t_2 whose fine grained elements are identified by the sets $\{P_k^{(1)}\} \equiv \{P_{\delta_1 \cup \delta_2}, P_C\}$ and $\{P_j^{(2)}\} \equiv \{P_{\Delta_j}\}$, respectively, are decoherent.

As the reader has certainly, grasped, the DH approach allows to claim, by making reference to $\text{F}_{\text{AM}}^{(2\text{slits}, t_1)}$, that the particle either goes through slit 1 or through slit 2 or it is absorbed. Similarly, by making reference to $\text{FAM}^{(2\text{slits};\, t_1,\, t_2)}$, one can state that the particle is either absorbed at time t_1 or it is, at time t_2, within one of the intervals Δ_j. But the fact that $\text{FAM}^{(2\text{slits, 2times})}$ is not a decoherent family, does not allow to make statements of the kind: the particle is now within the interval Δ_j and it went through slit 1 (slit 2). We stress that the just outlined distinction between $\text{FAM}^{(2\text{slits},\, t_1)}$ and $\text{FAM}^{(2\text{slits};\, t_1,\, t_2)}$ is an instance of the "single family rule," i.e., of the fact that quantum reasoning must always deal with a single decoherent family.

3. THE LOGIC OF DECOHERENT FAMILIES

In this section we discuss the way in which some of the proponents of the DH approach have suggested to give a precise and (as far as possible) classical logical structure to the theory. This is done by equipping the histories of any decoherent family with a Boolean structure and by defining (taking into account the probabilistic nature of the scheme) the concept of logical implication between histories of a decoherent family.

3.1. Decoherent Families and Boolean Algebras

Omnès,[9] and subsequently Griffiths,[7, 8] have shown that any family of histories is naturally equipped with a Boolean algebraic structure. The extreme relevance of such a step for the DH approach derives from the fact that it is a condition which should allow to recover classical logic and classical reasoning within a quantum context, avoiding in this way the problems arising in connection with quantum logic. Obviously, even though one can, formally, give an algebraic Boolean structure to any family, only in the case of decoherent families the probabilities defined by Eq. (2.4) define a classical probability measure on the set of histories, and, as such, they allow to give a physical meaning to the histories themselves.

Here we have no need to discuss this problem in its full generality; for our purposes it will be sufficient to consider a family characterized by a unique time instant t:

$$\text{HIS}^{(\alpha)} = \{(Q_\alpha, t)\} \tag{3.1}$$

where $\{Q_\alpha\}$ is the set of projection operators specifying the histories of the family. In this simplified case the conjunction and disjunction of two histories, and the negation of a given history, i.e., the logical connectives necessary to build a Boolean algebra, are defined according to:

$$\begin{aligned} \text{HIS}^{(\alpha)} \wedge \text{HIS}^{(\beta)} &= \{(Q_\alpha \wedge Q_\beta, t)\}, & Q_\alpha \wedge Q_\beta &= Q_\alpha Q_\beta \\ \text{HIS}^{(\alpha)} \vee \text{HIS}^{(\beta)} &= \{(Q_\alpha \vee Q_\beta, t)\}, & Q_\alpha \vee Q_\beta &= Q_\alpha + Q_\beta - Q_\alpha Q_\beta \\ \text{HIS}^{(\alpha)\perp} &= \{(Q_\alpha^\perp, t)\}, & Q_\alpha^\perp &= 1 - Q_\alpha \end{aligned} \tag{3.2}$$

It is an elementary task to show that all conditions characterizing a Boolean algebra are satisfied.

3.2. The Logical Implication Between Histories

As already stated, to work out a consistent "quantum reasoning" referring to histories one has to introduce in a mathematically precise way the idea of one history of a decoherent family implying (a fact we will denote by the symbol "⇒") another history of the same family. This is easily done by following the standard way used within a probabilistic context:

$$\text{HIS}^{(\alpha)} \Rightarrow \text{HIS}^{(\beta)} \qquad \text{if and only if} \quad \frac{p[\text{HIS}^{(\alpha)} \wedge \text{HIS}^{(\beta)}]}{p[\text{HIS}^{(\alpha)}]} = 1 \tag{3.3}$$

For future purposes, we call the attention of the reader on the fact that the above definition makes exclusive reference to the probabilities that the formalism attaches to the decoherent histories.

3.3. An Illuminating Discussion

When one adopts the above position about logical implication one is immediately confronted with a problem which can be summarized as follows: the same history, when considered as belonging to different decoherent families, can lead to contradictory implications. In fact, it is easy to show that one can identify two compatible histories[8] $\text{HIS}^{(\alpha)}$ and $\text{HIS}^{(\beta)}$ belonging to a certain decoherent family FAM_1, such that $\text{HIS}^{(\alpha)}$ implies $\text{HIS}^{(\beta)}$. Furthermore one can also identify a second decoherent family $\text{FAM}_2 \neq \text{FAM}_1$ such that the original history $\text{HIS}^{(\alpha)}$ and another history $\text{HIS}^{(\gamma)}$ belong to it and, within FAM_2, one can prove that $\text{HIS}^{(\alpha)}$ implies $\text{HIS}^{(\gamma)}$. We then have:

$$\text{HIS}^{(\alpha)}, \text{HIS}^{(\beta)} \in \text{FAM}_1; \qquad \text{HIS}^{(\alpha)} \Rightarrow \text{HIS}^{(\beta)} \tag{3.4}$$

$$\text{HIS}^{(\alpha)}, \text{HIS}^{(\gamma)} \in \text{FAM}_2; \qquad \text{HIS}^{(\alpha)} \Rightarrow \text{HIS}^{(\gamma)} \tag{3.5}$$

[8] I.e., such that there is a decoherent family to which they both belong.

Given the above facts one could naturally be tempted to conclude that $\text{His}^{(\alpha)}$ implies *both* $\text{His}^{(\beta)}$ *and* $\text{His}^{(\gamma)}$:

$$\text{His}^{(\alpha)} \Rightarrow [\text{His}^{(\beta)} \wedge \text{His}^{(\gamma)}] \tag{3.6}$$

proving in this way the inconsistency of the theory, since one can easily devise histories satisfying the above conditions but such that $\text{His}^{(\beta)}$ and $\text{His}^{(\gamma)}$ make reference to physical properties which cannot be simultaneously true. Formally, $\text{His}^{(\beta)}$ is associated to a certain projection operator B while $\text{His}^{(\gamma)}$ is associated to another such operator C, where $BC = CB = 0$, but $B \neq 1 - C$. Examples of this type have been considered for the first time by Aharonov and Vaidman[19] and subsequently by Griffiths and Hartle[20] in connection with Kent's criticism[21, 22] of the DH-approach.

A way to circumvent the just outlined difficulty has been suggested by Griffiths himself who, in his most recent papers,[7, 8] has repeatedly stressed the fundamental relevance of the "**single family rule**" (or logic, or framework) we have mentioned previously:[8]

> From now on we shall adopt the following as the fundamental principle of quantum reasoning: *A meaningful description of a (closed) quantum-mechanical system, including its time development, must employ a single framework.*

It seems to us that the most obvious way in which one has to understand the above rule is that when one resorts to logical tools to draw conclusions about histories he has to deal only with histories belonging to the same decoherent family. If one agrees on this point, to state that "$\text{His}^{(\alpha)}$ implies *both* $\text{His}^{(\beta)}$ *and* $\text{His}^{(\gamma)}$" is incorrect, since $\text{His}^{(\beta)}$ and $\text{His}^{(\gamma)}$ belong to incompatible families. Stated differently, since there is no decoherent family which can accommodate both of them, there is no consistent way of attaching a meaning to the conjunction of the two histories.

In our opinion, the fact that, while one can claim that $\text{His}^{(\beta)}$ and $\text{His}^{(\gamma)}$ are *separately* true (given for granted the truth of $\text{His}^{(\alpha)}$) it is nevertheless meaningless to consider them *together*, is rather puzzling. How can one accept this peculiar aspect of the formalism? The situation becomes even more embarrassing when one takes into account that the two histories $\text{His}^{(\beta)}$ and $\text{His}^{(\gamma)}$ may very well (as we have already said) make claims about properties which are mutually incompatible. Summarizing, we are inclined to agree with d'Espagnat when he claims that the situation we have just analyzed represents a real logical paradox.[16] In spite of the above remarks we are perfectly aware that, at the *purely formal* level, one can claim that the DH approach, when enriched by the single family rule interpreted in the above sense, is free from the just mentioned inconsistencies.

4. HOW MANY DECOHERENT FAMILIES ARE POSSIBLE?

As mentioned in the introduction, there is a serious source of difficulties in the theory, coming from the fact that there are many decoherent families—actually, most of them—which are manifestly devoid of any physical meaning. For example, it is easy to identify (and Griffiths himself has presented[7] an explicit example of such a situation) decoherent families in which Schrödinger's cat is not *dead or alive*, but *dead* + *alive*. Actually, Giardina and Rimini[23] have proved that starting from a quasi-classical decoherent family it is possible to build infinitely many inequivalent decoherent families which are, in general, highly non-classical. A question then naturally arises: which physical interpretation can be given to this type of decoherent families?

The proponents and the supporters of the decoherent histories approach have already identified this problem and have taken various positions about it. For example **Griffiths** has repeatedly stated[8] that *all decoherent families have to be put on the same footing*. In his words:

> The formalism allows the physicist to choose from a very large number of alternative frameworks or consistent families of histories, all of which are considered "equally valid" in the sense that no fundamental physical law determines which family should be used in any given case.

Accordingly, facing the problem of the existence of what we have characterized as *physically senseless decoherent families* he stated:[7]

> What would happen if, ten minutes from now, we were to abandon the quasi-classical framework for one in which, say, there is a coherent quantum superposition of the computer in distinct macroscopic states? Of course, nothing particular would happen to anything inside the box; we, on the other hand, would no longer be able to describe the object in the box as a computer, because the language consistent with such a description would be incompatible with the framework we were using for our discussion.

In previous papers,[4] he had also stated:

> Such histories [i.e., the senseless ones] do exist, and the consistent histories approach will assign probabilities to consistent families of **grotesque** histories, if that is what interests the theoretician. The point we wish to make is not that **grotesque** events are somehow ruled out by the consistent histories approach (obviously they are not), but simply that they are not an essential part of interpreting what happens in an **ordinary** consistent history.

In our opinion Griffiths' reasoning is shifty and weak: what is an **ordinary** family? What distinguishes it from the other families? Why should a theoretician consider only ordinary families? Alternatively, if he wants to consider also non-ordinary families, which physical meaning will he attach to them? Such fundamental questions are left unanswered.

Omnès[10] has made a serious attempt to give a precise meaning to the idea of a sensible decoherent family stating explicitly:

> One must first restrict oneself to a special class of logics [i.e., decoherent families]: those containing all the actual facts, i.e., all the real classical phenomena ... they may be said to be sensible.
>
> One can then assert what should be true. To begin with, actual facts will be taken to be true. Some other properties ... will also said to be true when they satisfy the two following criteria:
>
> — One can add them to *any* sensible logic while preserving consistency.
>
> — In all these augmented logics, (they are) logically equivalent to a factual phenomenon.

In this way he pretends to prove that only decoherent histories referring to definite macroscopic properties (past, present and future), as well as all histories referring to the properties of microsystems subjected to measurement processes are true. Thus, Omnès does not commit himself about the truth or falsity of the histories of all decoherent families, but only of those histories which are related to "classical" phenomena in accordance with the above criteria. However, Dowker and Kent[15] have shown that his argument is not satisfactory.

Gell-Mann and Hartle[12] proposed to consider an appropriate *measure of classicity* as a basic criterion to pick out the most classical family (or families) among all the decoherent ones:

> The impression that there is something like a classical domain suggests that we try to define quasiclassical domains precisely by searching for a measure of classicity for each of the maximal sets of alternative decohering histories and concentrating on the one (or ones) with maximal classicity.

Unfortunately, as they admitted:[13]

> ... we have not so far progressed beyond as to how that could be done.

Moreover, they have not made sufficiently clear the role which has to be given to the quasi-classical families which differ from the ones we experience. Natural questions arise: have such families to be considered as meaningless or do they correspond to other aspects of reality to which we have no access? It seems that these authors prefer this second alternative:[24]

> Among all the possible sets of alternative histories for which probabilities are predicted by the quantum mechanics of the universe, those describing a quasiclassical realm [i.e., quasiclassical decoherent families] like the one that includes familiar experience, are of special importance ... Such quasiclassical realms are important for at least two reasons ... [The second one being that] coarse grainings of this

usual quasiclassical realm are what we (humans and many other systems) use in the process of gathering information about the universe and making predictions about its future.

In our opinion one should put forward reasons to explain why we human beings do not perceive the existence of other domains. Gell-Mann and Hartle propose(12) the following explanation:

> If there are many essentially inequivalent quasiclassical domains, then we could adopt a subjective point of view, as in some traditional discussions of quantum mechanics, and say that the IGUS[9] "chooses" its coarse-graining of histories and, therefore, "chooses" a particular quasiclassical domain, or a subset of such domains, or further coarse grainings. It would be better, however, to say that the IGUS evolves to exploit a particular quasiclassical domain or set of such domains.

In view of the fact that the possibility of limiting the set of allowed families has already been entertained even by some of the supporters of the DH approach, it will be quite natural to take the same position to overcome the problems we are going to focus in the second part of this paper.

PART II

As already anticipated, this part will be devoted to a critical investigation of the nature of the probabilistic statements of Decoherent Histories. A nice way to stress this crucial point derives from taking into consideration the sharp remarks by J. S. Bell(2) about the ortodox interpretation of Quantum Mechanics:

> In the beginning, Schrödinger tried to interpret his wavefunction as giving somehow the density of stuff of which the world is made. He tried to think of an electron as represented by a wavepacket ... a wevefunction appreciably different from zero only over a small region in space. The extension of that region he thought of as the actual size of the electron ... his electron was a bit fuzzy. At first he thought that small wavepackets, evolving according to the Schrödinger equation, would remain small. But he was wrong. Wavepackets diffuse, and with the passage of time become indefinitely extended, according to the Schrödinger equation. But however far the wavefunction has extended, the reaction of a detector to an electron remains spotty. So Schrödinger's "realistic" interpretation of his wavefunction did not survive.
>
> Then came the Born interpretation. The wavefunction gives not the density of *stuff*, but gives rather (on squaring its modulus) the density of *probability*. Probability of what, exactly? Not of the electron *being* there, but of the electron being *found* there, if its position is "measured."

[9] An IGUS—Information Gathering and Utilizing System—is a complex adaptive system (as, for example, a human being) able to interact with the surrounding environment, gather and elaborate information coming from it.

> Why this aversion to "being" and insistence on "finding?" The founding fathers were unable to form a clear picture of things on the remote atomic scale. They became very aware of the intervening apparatus, and of the need for a "classical" base from which to intervene on the quantum system. And so the shifty split.

The proponents of the DH approach, in their first papers, seemed to share the worries so nicely expressed by Bell and to be mainly motivated by the desire to avoid giving any privileged status to measurement processes and observers. It is therefore interesting to discuss whether these requirements can be embodied within the probabilistic framework of Decoherent Histories. We begin by observing that Bell's pretension that the probabilities of a theory refer, e.g., to an electron "being" in a given place, is logically equivalent to requiring that statements like "the electron is confined within a given region" be either true or false independently of any further specification and of the fact that we only know how probable is that they are true.

We will now try to analyze in great detail the problem of the truth values of the probabilistic statements of the theory.

5. PROBABILITIES AND TRUTH-VALUES

As we have already said, the basic assumptions of the DH approach is that, given a family of histories satisfying the decoherence condition, the diagonal elements of the decoherence functional define a probability distibution on the histories of the considered family:

> Provided a consistency condition is satisfied, the corresponding Boolean algebra of events, called a framework, can be assigned *probabilities* in the usual way, and within a single framework quantum reasoning is identical to ordinary probabilistic reasoning. (Griffiths,[(7)])
>
> *Probabilities* (approximately obeying the rules of probability theory) can be assigned only to sets of histories that approximately decohere. (Gell-Mann and Hartle,[(12)])

But then the question put forward by John Bell arises: *probability of what, exactly*? Before giving an answer to this apparently elementary question let us comment in general on the meaning of probabilities in Classical and in Quantum Mechanics, by discussing the two following examples:

(a) Let us consider the tossing of a coin; the sample space has two elements, i.e., the two significative events "head" H and "tail" T. The probabilities assigned to these events are defined in accordance with the assumption that the coin is fair:

$$p(H) = 1/2, \qquad p(T) = 1/2 \tag{5.1}$$

These assignements define all statistical properties concerning the tossing of the coin.

(b) Let us now take into account the spin properties referring to the component along the z-axis of a spin 1/2 particle in the state $|x\uparrow\rangle$. Also in this case the sample space has two elements, which are the two physically meaningful events "spin up" $z\uparrow$ and "spin down" $z\downarrow$. The associated probabilities, in accordance with the quantum mechanical rules, are:

$$p(z\uparrow) = 1/2, \qquad p(z\downarrow) = 1/2 \tag{5.2}$$

Also in this case, the considered assignment characterizes all statistical properties concerning the spin of the particle along the z-axis.

It is obvious that, from the purely probabilistic points of view, the two cases are perfectly equivalent. *What is then the profound difference between Classical and Quantum Mechanics*? It resides uniquely in the radically different roles the two formalisms attach to probabilities. According to Classical Mechanics, the probabilities refer to objective properties (i.e., properties which are intrinsic to the system, assigned *a-priori* independently from any act of observation) of the physical systems under consideration, while Quantum probabilities refer exclusively to the possible outcomes of appropriate measurement procedures conditional under the measurements being performed. This means that physical properties do not exist by themselves, independently from measurement procedures. All this is well known, but the relevant question is: *is it possible to give a mathematically precise formal expression to this radical difference between the Classical and the Quantum cases*? The answer is also well known: the appropriate formal way to characterize the difference derives from the assignment of truth values to the various events. When we deal with classical probabilities, i.e., with probabilities referring to intrinsic, objective properties of physical systems, we can assign to each element of the sample space, and in general to each event of the corresponding Boolean algebra, a truth value:

In Classical Mechanics:

$$\exists h : \{x \in \text{Boolean Algebra of events}\} \rightarrow \{0, 1\} \tag{5.3}$$

Obviously, in general we do not know which truth value has to be attached to any event since, in the general case, we have only an incomplete knowledge of the actual physical situation. This fact, however, is totally irrelevant from the ontological point of view: if we believe that Classical probabilities refer to properties objectively possessed by physical systems, properties which exist independently from us and our knowledge, then we can legitimately state that any event (the mathematical counterpart of the physical properties) is associated to a precise truth value: it is "true" (1) when it corresponds to properties actually possessed by the physical system, it is "false" (0) if it describes properties which are not possessed by the physical system. Moreover, we can identify the essential features that the correspondence h must exhibit. Technically h must be a two-valued homomorphism from the Boolean Algebra formed by the subsets of the sample space onto the Boolean Algebra $\{0, 1\}$. This means that the application h must preserve the join, meet and complement relations:

$$\text{(i)} \quad h(x \vee y) = h(x) \vee h(y) \tag{5.4}$$

$$\text{(ii)} \quad h(x \wedge y) = h(x) \wedge h(y) \tag{5.5}$$

$$\text{(iii)} \quad h(x^{\perp}) = h(x)^{\perp} \tag{5.6}$$

for every x, y belonging to the Boolean Algebra. Obviously, such requests must be satisfied if we want to be allowed to use classical reasonings about classical events. For example, suppose x is a true event: $h(x) = 1$. Then, the classical way of reasoning requires its negation to be false: $h(x^{\perp}) = 0 = 1^{\perp} = h(x)^{\perp}$. Similarly, if x and y are two events which are true and false, respectively $-h(x) = 1$, $h(y) = 0$—then their join must be true and their meet must be false:

$$h(x \vee y) = 1 = 1 \vee 0 = h(x) \vee h(y) \tag{5.7}$$

$$h(x \wedge y) = 0 = 1 \wedge 0 = h(x) \wedge h(y) \tag{5.8}$$

and similarly for all other cases.

As we have stated, within Quantum Mechanics, systems do not possess objective properties and probabilities refer simply to measurement outcomes. In such a context, one cannot, in general, attribute *any* truth value to the events which are correlated with the "properties" (i.e., with the projection operators) of quantum systems:

In Quantum Mechanics:

$$\nexists h : \{x \in \text{Boolean Algebra of events}\} \to \{0, 1\} \tag{5.9}$$

After these remarks, we can come back to our analysis of Decoherent Histories; in this context, the elements of the sample space, and the events of the Boolean algebra associated to it, are the histories belonging to a single family, and, when the decoherence condition is satisfied, a probability distribution on such histories is defined. In this case too, when one limits his considerations to a single family of decoherent histories, the situation is, from a formal point of view, identical to the one of the Classical and of the Quantum cases. However, to have a complete picture one must give a precise answer to the original question: *the probabilities which the DH approach attaches to any decoherent history* (*which we will denote as DH-probabilities*) *what do they refer to*? To properties objectively possessed by physical systems or to measurement outcomes, just as the Quantum probabilities? Obviously, since the purpose of the DH-approach is that of solving the interpretative difficulties of Quantum Mechanics and to put forward a realistic theory, the DH-probabilities, just like Classical ones, must refer to objective and intrinsic properties of physical systems: there is no other reasonable alternative. But, if this is the situation, if these are the aims of lithe theory, our previous analysis leads to conclude that any decoherent history must have a precise truth value, independently from the fact that, in general, we have only a probabilistic knowledge about the system. Moreover, the relation between histories and truth values must be a homomorphism:

$$\text{DH-probabilities like Classical probabilities} \Rightarrow$$

$$\exists h : \{\textsc{His} \in \text{Decoherent } \textsc{Fam}\} \to \{0, 1\},\ h \text{ is a homomorphism} \qquad (5.10)$$

In the second section of this paper we have mentioned that, within the DH-approach any reasoning, any logical argument, must be entirely developed within the context of a single family of decoherent histories. If this rule is not respected one risks to draw contradictory conclusions. At first sight, this feature of the theory *might* be considered as implying that every decoherent family has its precise homomorphism, independently from the truth values attached to histories belonging to decoherent families which are incompatible with the previous one. If this is the situation, it *would* be more appropriate to denote the homomorphism as $h_{\textsc{Fam}}(\textsc{His})$. This notation makes evident the fact that classical reasoning is valid **iff** it is confined to a single decoherent family, and, as such, it depends crucially on the family we are dealing with. To better grasp this important point one could make reference, once more, to the histories about the two slits experiment discussed in Section 2. In such a case with reference, e.g., to $\textsc{Fam}^{(2\text{slits},\, t_1)}$, one can state that the particle goes *either* through slit 1 *or* through slit 2 and that if it is true that it goes through slit 1 then it is false that it goes through slit 2 and viceversa. Similarly, the probabilistic structure of the set of histories of $\textsc{Fam}^{(2\text{slits};\, t_1,\, t_2)}$ allows one to claim that a particle which has not been absorbed by the screen, is objectively within one and only one of the disjoint intervals Δ_j, i.e., only one of these histories turns out to be true, all the remaining ones being false. In spite of this, as repeatedly stressed, statements asserting through which slit the particle went and which final position it has are not legitimate since such histories belong to the family $\text{FAM}^{(2\text{slits},\, 2\text{times})}$ which is not decoherent and as such no truth value can be attached to them.

To conclude this section, we consider it appropriate to analyze some statements by Griffiths about the truth values which one can attach to decoherent histories:(8)

> One important difference between Omnès and CHQR [Consistent Histories and Quantum Reasoning] is in the definition of "true." In CHQR, "true" is interpreted as "probability one." Thus if certain data are assumed to be true, and the probability, conditioned upon these data, of a certain proposition is one, then this proposition is true. The advantage of this approach is that as long as one sticks to a single framework, "true" functions in essentially the same way as in ordinary logic and probability theory. However, because probabilities can only be discussed within some framework, comparison of "true" between incompatible frameworks are impossible, and in this sense "true" interpreted as "probability one" must be understood as relative to a framework.
>
> The feature just mentioned has been criticized by d'Espagnat. But it is hard to see how to get around it if one wishes to maintain (as do Omnès and I) that reasoning inside a single framework should follow classical rules, and classical rules associate "true" (in a probabilistic theory) with "probability one."

Such statements seems to us to confuse the reader rather than clarifying the matter. In fact:

- In the usual physical language the statement that "something is true" makes reference to *objective* properties of a physical system, to some *elements of physical reality*, in Einstein's language. If Griffiths resorts to the expression "true" having in mind something different he should be very precise about it, or even resort to a different term.

- In Classical Statistical Mechanics, while "true" is correctly associated with "probability one," it is not the case that "true" is associated *only* with "probability one." Within such a theory *any* proposition has a precise truth value quite *independently* from the probability which is attached to it.[10] As a consequence, when Griffiths writes: *in CHQR*, "*true*" *is interpreted as* "*probability one*," he must choose one and only one of the two following alternatives:

1. Just as in the case of Classical Statistical mechanics, if the theory associates to a proposition probability one (zero) then the proposition is certainly true (false), while in all other instances *the proposition has in any case a truth value*, *which*, *however*, *is unknown to us*. If this is the case then *every decoherent history has a truth value* (which, as previously discussed

[10] Exception made for the fact that, if its probability equals one, then the proposition must obviously be true, while if it equals zero, then it must be false.

can be formally described by an appropriate homomorphism). We will analyse in what follows the implications of taking such a position.

2. *Only the decoherent histories having probability one are true and those having probability zero are false, while all other histories do not possess a truth value.* If this is the case, then Griffiths is discriminating some decoherent histories from the remaining ones and he is giving a particular conceptual status only to some of them. Some histories have a precise physical meaning and ontological status, since they have a truth value, while all the remaining ones have no physical meaning whatsoever. Such a position is perfectly legitimate; however we cannot avoid remarking that "in the real world" probabilities which are exactly equal to one never occur.[11] Accordingly, there is always a precise (possibly very small) probability that different things might happen. For instance, the wavefunctions of all physical systems have non compact support in configuration space (exception made for single time instants). Consequently, a physical system which is at a certain place at a given instant can be, at any subsequent time, at an arbitrarily far away position (even though the probability of such an event is extremely small in the case of macroscopic objects). In simpler terms, the fact that a table is here now does not imply, with probability equal to one, that it will still be here within a second. The conclusion seem to us unavoidable: if this is what Griffiths has in mind, then *nothing is true and nothing is false*, since nothing has probability one or zero of occurring.

Moreover, within Standard Quantum Mechanics itself it is perfectly legitimate and consistent to attach a definite truth value to events (or equivalently to histories) which have probability one or zero to actually occur. There is no need to resort to a new interpretation, such as the one characterizing Decoherent Histories, to get this result.

6. THE "SINGLE FAMILY" RULE AND TRUTH-VALUES

In the previous sections of this work we have mentioned that a basic assumption of the DH approach of Griffiths is the strict request that any argument must be developed within a *single family* of decoherent histories. If this rule is not respected one risks to derive (uncorrectly) contradictory and/or inconsistent conclusions. Griffiths himself, in his most recent papers,[7, 8] has repeatedly stressed the fundamental relevance of the "single family rule."

[11] Leaving aside some pathological and physically uninteresting cases.

In this section we will make an attempt to understand what he means exactly by requiring that the "single family rule" be an essential part of the theory, and in which cases it must be applied. To clarify this point, we begin by quoting a long sentence from a paper[5] published in 1987 by this author:

> A sheet of paper weighing 5 g is torn in two, the two pieces are placed in opaque envelopes, and one is mailed to a scientist in London and the other to a scientist in Paris.
>
> On the basis of this information, the masses of the individual pieces of paper are unknown but strongly correlated in that their sum must be 5 g. Thus, for example, if the envelope in London is opened and the piece of paper is weighed and has mass of 3 g, we can at once conclude that the piece in Paris has mass of 2 g ...
>
> Needless to say, the fact that one can immediately infer from a measurement in London the weight of a piece of paper hundreds of kilometers away in Paris has nothing to do with some strange "action at a distance" ... Weighing the piece in London has no *physical* effect upon the piece of paper in Paris. What it does affect is our *knowledge* about the latter, which is something very different. Of course, if we fail to distinguish between what is there and our knowledge of what is there, we may be confused into thinking that measurement on one object have "spooky" effects on a very distant object.

In this passage Griffiths identifies correctly two different levels of reasoning within the context of Classical Physics:

- On one side there is the **knowledge we have about a given physical system**; such a knowledge is *subjective* since it depends crucially from the information we have and from the aspects of the system we are interested in considering. Moreover, it has, typically, a *probabilistic* nature, since (in practice) it is impossible to determine with infinite precision all variables which characterize the state of the system. In the specific example chosen by Griffiths, the knowledge by the observer exhibits also some "non-local" features, since it implies an instantaneous change of our knowledge about far away systems.

- On the other side we have the **properties objectively possessed by physical systems**. They have an *objective* status, since they do not depend on us and on our acting (such as performing a measurement process) on the physical system: accordingly, they exist *a priori*. Moreover, with reference to the example considered by Griffiths there are no instantaneous changes of properties at-a-distance.

Obviously, within standard Quantum Mechanics, the distinction we have made is meaningless simply because within such, a framework there are no properties which are objective and independent from the measurement processes devised to ascertain them. On the contrary, the above distinction is perfectly legitimate within Classical Mechanics, and even within the DH approach, if it is true that the DH-probabilities, just as the classical ones, describe objective properties of physical systems.

Let us come back to the "single family" rule: it represents, as explicitly pointed out by Griffiths, *the fundamental rule of quantum reasoning*. But let us raise the question: what precisely is "*quantum reasoning*"? Griffiths writes:[7]

> The type of quantum reasoning we shall focus on this section is that in which one starts with some information about a system, known or assumed to be true, and from these *initial data* tries to reach valid *conclusions* which will be true if the initial data are correct ...
>
> Since quantum mechanics is a stochastic theory, the initial data and the final conclusions will in general be expressed in form of probabilities, and rules of reasoning are rules for deducing probabilities from probabilities ..
>
> Since probabilities in ordinary probability theory always refer to some sample space, we must embed quantum probabilities referring to properties or the time development of a quantum system in an appropriate framework.

In ref. 8 he adds:

> Choosing a coarse-graining [of phase-space] and choosing a quantum framework are analogous in that in both cases the choice is one made by the physicist in terms of the physical problems he wishes to discuss ... this choice has no influence on the behavior of the system being described, although it may very well limit the type of description that can be constructed.

These quotations, as well as many others, show that what Griffiths denotes as "quantum reasoning," and then also the "single family rule," make exclusive reference to our knowledge about a physical system and to the way in which we can take advantage of such a knowledge to derive new information about it. However, in our opinion, there is a fundamental point which remains obscure: *does the "single family rule" refer exclusively to the knowledge we have about physical systems and to the way one can take advantage of such a knowledge to infer further information about them, or does it refer also to the properties which are objectively possessed by them*? Since in the very formulation of the rule no reference is made to the just raised question, let us analyze the two possible scenarios we have outlined.

- Let us suppose that the "single family rule" applies not only to our knowledge about physical systems but also to the properties characterizing them. Let us then consider a history $\text{His}^{(\alpha)}$ referring to the properties of an elementary particle, such as "its spin points in a certain direction" (obviously analogous considerations can be developed with reference to macroscopic objects). What can be stated about the truth value of such a history?

1. The probability associated to the history (as for all actual histories) is neither identically one, nor identically zero. In accordance with the previous discussion Griffiths has not been explicit about whether $\text{His}^{(\alpha)}$ has a truth value or not. If it does not the history represents simply a Statement devoid of any sense. In the case of microsystems this could be accepted, but, as already remarked, just the same situation can occur for macroscopic systems. In brief the theory cannot make any interesting statement. Let us then analyze the other possibility, i.e., that the history has a truth value and that it is, e.g., true.

2. Since, in agreement with the position we are analyzing, the "single family" rule claims that the properties of physical systems are unavoidably linked to the decoherent family we use to describe them, the history $\text{His}^{(\alpha)}$ does not have any truth value *per se*, but it acquires it only with reference to the decoherent family we decide to choose. Such a situation is unprecedented in classical physical theories where the properties of physical systems are "objective" and cannot depend on further specifications.

3. However, a further crucial fact has to be taken into account. As remarked in the introduction, the same history belongs, in general, to different and incompatible decoherent families. Accordingly, let us consider now a second decoherent family, which is incompatible with the previous one, but containing $\text{His}^{(\alpha)}$. Since we have changed the family, the "single family" rule (according to the interpretation we are discussing), forbids us to claim that such a history is necessarily still true, since such a conclusion was valid within another family. We are, in a sense, back to square one, just because we have changed the family. Consequently, the history $\text{His}^{(\alpha)}$, when considered as a member of the second family, might have a different truth value, it might be false.[12] The fact that the truth value of a history and accordingly also the properties of physical systems may change by changing the decoherent family one decides to take into account does not represent by itself an unacceptably peculiar feature of the formalism?

Accepting that the truth value of a precise history belonging to a decoherent family (and consequently the assertions about properties objectively possessed by individual physical systems), could change according to the decoherent family we choose to describe them, would represent, within the present formalism, the exact analogue of accepting the contextual nature of most properties in hidden variable theories (recall that resorting

[12] If one accepts that the truth value cannot change when the family is changed one accepts one of the assumptions we will consider below and which we will show to clash with the other assumptions we will put forward.

to contextuality is the appropriate way for such schemes to circumvent the Kochen and Specker no-go theorem[(25)]). We are not worried by the fact that some properties of a physical system could turn out to be contextual, i.e., non objective, since they depend upon the overall context.

However, it has to be stressed that while contextuality can be accepted within hidden variable theories in which it simply means that the outcomes of measurements depend, besides the "state" characterizing the system, also on the way the measurement is carried on,[13] the situation in the present case is radically different. Here the dependence from the context does not refer to *different actual situations* but to *different choices* about what we want to assert about our physical system. It seems to us that accepting such a form of contextuality within the DH approach would spoil the theory not only of its original meaning, but actually of any meaning at all. In fact within DH, and contrary to the case of hidden variable theories, there is no way to divide properties into noncontextual and contextual, so that; if one gives up the assumption that the truth value of a given history does not depend from the perspective we choose to speak about the system, one must consider all properties as contextual. On the other hand, as Dürr, Goldstein and Zanghí[(27)] have appropriately stressed:

> Properties that are merely contextual are not properties at all; they do not exist, and their failure to do so is in the strongest sense possible.

• Let us suppose now that the "single family rule" refers only to our knowledge of physical systems and to the way we can make use of such a knowledge to get new information about them. In this case, it is obvious why "quantum reasoning" turns out to have, in general, a probabilistic nature and why it is subjective, (in the sense that it depends in a crucial manner from the information we have and from what we are interested in knowing), but these facts do not, in any way and as it happens within a classical context, make subjective also the physical reality itself. All these facts seem to find a confirmation in some statements by Griffiths himself:[(7)]

> A classical analogy, that of "coarse-graining" in classical statistical mechanics, is helpful in seeing why the physicist's freedom in choosing a quantum framework does not make quantum theory subjective, or imply that this choice influences physical reality. Coarse graining means dividing the classical phase-space into a series of cells of finite volume. From the point of view of classical mechanics, such a coarse-graining is, of course, arbitrary; cells are chosen because they are convenient for discussing certain problems, such as macroscopic (thermodynamic)

[13] Equivalently, one could claim, with Bell,[(26)] Dürr, Goldstein and Zanghí,[(27)] that in hidden variable theories, typically in Bohmian mechanics, the appropriate way out from the necessity of accepting contextuality derives from claiming that what the theory is about is simply the positions of all particles of the universe at any time.

irreversibility. But this does not make classical statistical mechanics a subjective theory. And, in addition, no-one would ever suppose that by choosing a particular coarse-graining, the theoretical physicist is somehow influencing the system.

As we have claimed various times, within Classical Mechanics there are two levels of reasoning: the first one makes deference to our knowledge about a physical system, and the second one to the properties objectively possessed by the system and to their reality, quite independently from our knowledge about them. With reference to the previously analyzed example of tossing a coin we can state:

(a) Our **knowledge** about the outcome is that there is a probability 1/2 of getting *heads* and a probability 1/2 of getting *tails*: this is all what we know.

(b) However, we can also argue in terms of **objectively possessed properties**, and we can legitimately claim that the outcome will surely be either *heads* or *tails* and we can also claim that in the case in which the outcome will be *heads*, then certainly it will not be *tails* (this being simply a more lengthy way to state that the truth values which are attached to all possible events concerning the tossing of a coin form a homomorphism). These statements might seem rather trivial and uninteresting, but the perspective changes radically, as implied by the Kochen and. Specker theorem, when one takes into account that they refer not to classical but to quantum properties.

Accordingly, if one takes this position about the "single family" rule, the truth value of the decoherent histories is determined once for all and it does not depend on the particular family we use to describe such properties. It is only our (probabilistic) way of reasoning which must be confined to a single family of decoherent histories. This means that it is possible to choose the homomorphisms associated to the various families in such a way that:

$$h_{\text{FAM}}(\text{HIS}^{(\alpha)}) = \text{const.}, \qquad \forall \text{ decoherent FAM}, \text{HIS}^{(\alpha)} \in \text{FAM} \tag{6.1}$$

In our opinion, this is the only reasonable way to interpret the "single family rule."

6.1. More About the "Single Family Rule"

At this point, it could be useful to stress the basic difference between the problem discussed in the example of Section 3.3, which is considered as irrelevant by the supporters of the DH, and the question raised in the previous paragraph, i.e., whether or not the truth value of a given history depends on the family to which it belongs. In the above mentioned example, one tries to derive a contradiction by taking into account the conjunction of *different histories* belonging to different and incompatible families; on the contrary, in the previous paragraph we have considered *the same history* even though we view it as a member of different decoherent families. Moreover, the argument of the example is entirely based on the use of the logical implication "$\Rightarrow$" and, consequently, it deals with the *probabilities* which characterize the various decoherent histories, with the (probabilistic) *knowledge* we have about the system and its properties. On the contrary, the arguments of the previous section call into play only the *truth values* of the histories: in other words, we have argued only with reference to the *objective properties* possessed by physical systems, quite independently from our knowledge about them.

We can make more understandable the difference between the two arguments by resorting to the logical analysis of DH. Fundamentally, following the previous example, one would be tempted to claim that:

$$\{\text{H}{\small\text{IS}}^{(\alpha)} \Rightarrow \text{H}{\small\text{IS}}^{(\beta)}, \text{H}{\small\text{IS}}^{(\alpha)} \Rightarrow \text{H}{\small\text{IS}}^{(\gamma)}\} \supset \{\text{H}{\small\text{IS}}^{(\alpha)} \Rightarrow [\text{H}{\small\text{IS}}^{(\beta)} \wedge \text{H}{\small\text{IS}}^{(\gamma)}]\} \tag{6.2}$$

However, since $\text{H}{\small\text{IS}}^{(\beta)} \wedge \text{H}{\small\text{IS}}^{(\gamma)}$ *does not belong to any decoherent family*, it is not given any probability to it. Correspondingly the logical implication:

$$\text{H}{\small\text{IS}}^{(\alpha)} \Rightarrow [\text{H}{\small\text{IS}}^{(\beta)} \wedge \text{H}{\small\text{IS}}^{(\gamma)}] \tag{6.3}$$

cannot even be defined. In our opinion this way of interpreting the "single family rule" is the only sensible one which can be accepted. On the other hand, in the previous section we have asked whether, since the *same* history can belong to different decoherent families and if in all such families it has a precise truth value:

$$\begin{aligned} \{\text{H}{\small\text{IS}}^{(\alpha)} \in \text{Decoherent F}{\small\text{AM}}_1\} &\supset \exists h_1 : h_1[\text{H}{\small\text{IS}}^{(\alpha)}] \Rightarrow \{0, 1\} \\ \{\text{H}{\small\text{IS}}^{(\alpha)} \in \text{Decoherent F}{\small\text{AM}}_2\} &\supset \exists h_2 : h_2[\text{H}{\small\text{IS}}^{(\alpha)}] \Rightarrow \{0, 1\} \end{aligned} \tag{6.4}$$

one should require that:

$$h_1[\text{H}{\small\text{IS}}^{(\alpha)}] = h_2[\text{H}{\small\text{IS}}^{(\alpha)}] \tag{6.5}$$

In our opinion, in no paper proposing the DH interpretation of Quantum mechanics a clear cut answer to this fundamental question has been given. In its formulation the "single family rule" is mute about this problem. This is why we have considered various possible alternative readings of the rule and we have called attention to their conceptual implications.

7. DECOHERENT HISTORIES AND THE KOCHEN AND SPECKER THEOREM

In previous sections we have analyzed some points of the DH approach which seem to us not to have been adequately clarified and which make to some extent problematic the interpretation of the theory. We have suggested what we consider the most obvious and natural position about it if it has to be taken as a serious alternative to the orthodox interpretation allowing to take a realistic position about natural processes, the principal aim of the proponents of this scheme. In particular we stress the following points:

1. The probabilities which the theory attaches to decoherent histories should have the same ontological status as the classical probabilities, i.e., they should refer to objective properties which may or may not be actually possessed by the physical system we are dealing with. Accordingly, for every decoherent family there should be an homomorphism from the Boolean set of its histories onto the Boolean set $\{0, 1\}$:

$$\textsc{Fam} \text{ is Decoherent} \Rightarrow \exists h_{\textsc{Fam}} : \textsc{Fam} \rightarrow \{0, 1\}$$

2. The homomorphism under 1 should depend on the history but not on the different decoherent families to which it may belong:

$$h_{\textsc{Fam}}(\textsc{His}^{(\alpha)}) = \text{const.} = h(\textsc{His}^{(\alpha)})$$
$$\forall \text{ Decoherent } \textsc{Fam} \text{ such that } \textsc{His}^{(\alpha)} \in \textsc{Fam}$$

i.e., the truth value of a history and consequently also the physical properties of which such history speaks, should be independent from the particular decoherent family one is dealing with. Here, then, we interpret the "single family" rule as referring only to our knowledge of physical systems, not to their properties, as we have discussed in Section 6.

In the following we shall prove that these two requirements cannot hold simultaneously, unless one drastically restricts the set of allowed Decoherent Families.

7.1. The Theorem

Consider the following four assumptions:

(a) Every decoherent family has a natural Boolean structure which, for one-time families, is defined as in Section 3.1.

(b) For every decoherent family there is an homomorphism from the Boolean set of its histories onto the Boolean set $\{0, 1\}$:

$$\text{Fam is Decoherent} \Rightarrow \exists h_{\text{Fam}}: \text{Fam} \rightarrow \{0, 1\}$$

(c) The homomorphism under (b) depends on the history but not on the different decoherent families to which it may belong:

$$h_{\text{Fam}}(\text{His}^{(\alpha)}) = \text{const.} = h(\text{His}^{(\alpha)})$$
$$\forall \text{ Decoherent Fam such that His}^{(\alpha)} \in \text{Fam}$$

(d) All decoherent families have the same status and must be taken into account;

We now prove that they lead to a contradiction. Before proceeding we would like to stress that what we are going to derive is a mathematical theorem which is logically implied by the just mentioned assumptions.[14]

Consider a set D-His of histories (which belong to decoherent families) involving only one time instant[15] t:

$$\text{D-His} = \{\text{His} = (P, t),\ P \text{ is a projector operator, } t \text{ is a fixed time}\} \tag{7.1}$$

Recall that D-His contains the null history $(0, t)$ and the identity history $(1, t)$.

We now remark that, taking advantage of the Boolean character of every family, the set D-His can be naturally equipped with the structure of a partial Boolean algebra (PBA). A PBA is a set B with two distinguished elements 0 and 1, the meet ($\vee$), join ($\wedge$) and complement ($^{\perp}$) relations and a commensurability relation R satisfying the following axioms:

1. $\forall x \in B$: xRx
2. $\forall x, y \in B$: $xRy \Rightarrow yRx$
3. $\forall x \in B$: $0Rx$, $1Rx$

[14] Even though the above assumptions have implications concerning the meaning of the theory, they are formulated in a precise way. Therefore, if somebody, like Griffiths and the referee, believe that we are violating the "single family rule," he has to tell clearly which one (or ones) of the assumptions violates such a rule. After having shown that such a rule is violated and having identified the precise assumption which he thinks clashes with the rule, he should plainly declare that the accepts the consequences of this choice, consequences which we have analyzed in details in the previous sections.

[15] We recall that a one-time family contains histories of the type (P_k, t), t being an arbitrary time instant and the operators $\{P_k\}$ being an exhaustive and exclusive set summing up to the identity operator. Note that every one-time family is automatically decoherent.

4. $\forall x, y \in B$: $xRy \Rightarrow xRy^{\perp}$

5. $\forall x, y, z \in B$: $(xRy, xRz, yRz) \Rightarrow xR(y \vee z), xR(y \wedge z)$

6. $\forall x \in B$: $x \vee x = x$

7. $\forall x \in B$: $0 \vee x = x \vee 0 = x, 1 \wedge x = x \wedge 1 = x$

8. $\forall x \in B$: $x \wedge x^{\perp} = 0, x \vee x^{\perp} = 1$

9. $\forall x, y, x \in B$: (xRy, xRz, yRz)

$$\Rightarrow x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \tag{7.2}$$

The commensurability relation can be easily defined: we say that two histories $\text{His}^{(1)}$ and $\text{His}^{(2)}$ are commensurable **iff** they are compatible, i.e., they belong together to at least one decoherent family. Note that since in our proof we are considering only one-event histories, compatibility reduces to commutativity of the projection operators appearing in the two histories:

$$\text{His}^{(1)} = (P_1, t); \qquad \text{His}^{(2)} = (P_2, t); \qquad \text{His}^{(1)} \, R \, \text{His}^{(2)} \Leftrightarrow [P_1, P_2] = 0 \tag{7.3}$$

Moreover, the meet, join and complement operations are defined like in Section 3. It is now immediate to check that if $[P_1, P_2] = 0$ then the first 5 axioms (7.2) are satisfied. Also the remaining axioms are easily checked to hold. Just to give an example we remark that the last axiom requires the distributivity property to be satisfied only for triplets of elements which are commensurable with each other, i.e., for triplets of histories belonging to the same decoherent family. On the other hand we know that in a decoherent family (due to its Boolean structure) the distributive property holds.

The second step is to remark that due to the fact that for any decoherent family there exists a two-valued homomorphism (assumption b) such that the image does not depend on the family (assumption c), we can also define in D-His a two-valued homomorphism H. We recall that a homomorphism h between two PBAs, B and C is a relation satisfying the following properties:

1. $\forall x, y \in B$: $xRy \Rightarrow h(x) \, Rh(y)$ (7.4)

2. $\forall x \in B$: $h(x^{\perp}) = [h(x)]^{\perp}$ (7.5)

3. $\forall x, y \in B$: $xRy \Rightarrow h(x \wedge y) = h(x) \wedge h(y), h(x \vee y) = h(x) \vee h(y)$ (7.6)

Now we can go on by defining the homomorphism H according to:

$$H\colon \text{D-IS} \rightarrow \{0, 1\} \qquad \text{by putting} \quad H(\text{HIS}) = h(\text{HIS}) \tag{7.7}$$

where $h(\text{HIS})$ is the homomorphism discussed in Section 5 and 6. It is immediate to check that H has all the required properties. It is important to stress that H is unambiguously and correctly defined just because the homomorphism h depends only on the histories and not on the families, i.e., any history has a precise truth-value, 0 or 1. If $h(\text{HIS})$ were family-dependent it would not be guaranteed that one could define H for the one time histories.

Now we remark that D-HIS is isomorphic to $P(\mathscr{H})$, the set of all projection operators of the Hilbert space $\mathscr{H}$ (which, in turn, is a PBA, the commensurability relation being, once more, commutativity between projection operators). The isomorphism is obviously given by the correspondence:

$$\Phi\colon \text{D-HIS} \rightarrow P(\mathscr{H})$$

$$\text{HIS} \equiv (P, t) \Leftrightarrow \Phi(\text{HIS}) = P \tag{7.8}$$

Once we have shown that D-HIS and $P(\mathscr{H})$ are isomorphic, we can "carry" the two-valued homomorphism H from D-HIS to $P(\mathscr{H})$ as follows:[16]

$$K\colon P(\mathscr{H}) \rightarrow \{0, 1\}, \qquad P \rightarrow H[\Phi^{-1}(P)] \tag{7.9}$$

Let us summarize the situation. We have proved that if conditions (a)–(c) hold, then one can define a homomorphism between the *PBA* $P(\mathscr{H})$ of the projection operators on $\mathscr{H}$ which are associated to histories of the set D-HIS and the Boolean set $\{0, 1\}$. Assumption (d) of the theorem amounts to claim that the just considered PBA includes all projections operators on $\mathscr{H}$. But Kochen and Specker, in their celebrated paper,[(25)] have proved precisely that such a homomorphism cannot exist. Thus we have proved[17] that one cannot add to assumptions (a)–(c) we have put forward at the beginning of this Section, and which according to us are necessary ingredients to make the decoherent histories approach physically sensible, the request that all decoherent families have to be taken into account without meeting a contradiction.

[16] It is immediately checked that K is a homomorphism.

[17] We mention that a generic—taking few lines—statement about the possibility of working out such a proof appeared in a paper by S. Goldstein and D. N. Page.[(17)]

7.2. A Clarifying Example

We try to further clarify the situation by discussing the "paradigmatic" example of a spin 1 particle for which we limit our considerations only to the spin degrees of freedom; another example has appeared in ref. 28. Be $\Sigma_x^2, \Sigma_y^2, \Sigma_z^2$ the squares (in units of $\hbar^2$) of the spin components along three orthogonal directions x, y and z, respectively. The three considered operators commute and have a complete set of common eigenvectors which we will denote by $|\alpha\rangle$, $|\beta\rangle$ and $|\gamma\rangle$, satisfying:

$$\begin{array}{lll} \Sigma_x^2\,|\alpha\rangle=|\alpha\rangle & \Sigma_x^2\,|\beta\rangle=|\beta\rangle & \Sigma_x^2\,|\gamma\rangle=0 \\ \Sigma_y^2\,|\alpha\rangle=|\alpha\rangle & \Sigma_y^2\,|\beta\rangle=0 & \Sigma_y^2\,|\gamma\rangle=|\gamma\rangle \\ \Sigma_z^2\,|\alpha\rangle=0 & \Sigma_z^2\,|\beta\rangle=|\beta\rangle & \Sigma_y^2\,|\gamma\rangle=|\gamma\rangle \end{array}$$

If we consider the projection operators on the one-dimensional manifolds spanned by the above states ($P_\alpha=|\alpha\rangle\langle\alpha|$, $P_\beta=|\beta\rangle\langle\beta|$ and $P_\gamma=|\gamma\rangle\langle\gamma|$) we have

$$\Sigma_x^2=P_\alpha+P_\beta; \qquad \Sigma_y^2=P_\alpha+P_\gamma; \qquad \Sigma_z^2=P_\beta+P_\gamma \tag{7.10}$$

Let us now consider the one-time family $\text{Fam}^{(x,y,z)}$ whose maximally fine-grained histories are characterized by the projection operators $\{P_k^{(1)}\}\equiv\{P_\alpha, P_\beta, P_\gamma\}$, i.e., the histories:

$$\text{His}^{(\alpha)}=\{P_\alpha, t_1\}; \qquad \text{His}^{(\beta)}=\{P_\beta, t_1\}; \qquad \text{His}^{(\gamma)}=\{P_\gamma, t_1\} \tag{7.11}$$

where t_1 is a time instant following the initial time t_0 in which the spin state of the particle is, as usual, described by the statistical operator W.

We note that $\text{Fam}^{(x,y,z)}$ is automatically a decoherent family and that only one of the three considered histories $\text{His}^{(\alpha)}$, $\text{His}^{(\beta)}$ and $\text{His}^{(\gamma)}$ is true. This is due to the fact that the considered histories are mutually exclusive:

$$\text{His}^{(\alpha)}\wedge\text{His}^{(\beta)}=\text{His}^{(\alpha)}\wedge\text{His}^{(\gamma)}=\text{His}^{(\beta)}\wedge\text{His}^{(\gamma)}=0 \tag{7.12}$$

and exhaustive:

$$\text{His}^{(\alpha)}\vee\text{His}^{(\beta)}\vee\text{His}^{(\gamma)}=1 \tag{7.13}$$

Let us suppose now that $\text{His}^{(\alpha)}$ is true. Then:

$$h[\text{His}^{(\alpha)}]=1; \qquad h[\text{His}^{(\beta)}]=0; \qquad h[\text{His}^{(\gamma)}]=0 \tag{7.14}$$

This means that:

$$h[\Sigma_x^2] = h[\text{His}^{(\alpha)} \vee \text{His}^{(\beta)}] = h[\text{His}^{(\alpha)}] \vee h[\text{His}^{(\beta)}] = 1$$
$$h[\Sigma_y^2] = h[\text{His}^{(\alpha)} \vee \text{His}^{(\gamma)}] = h[\text{His}^{(\alpha)}] \vee h[\text{His}^{(\gamma)}] = 1 \qquad (7.15)$$
$$h[\Sigma_z^2] = h[\text{His}^{(\beta)} \vee \text{His}^{(\gamma)}] = h[\text{His}^{(\beta)}] \vee h[\text{His}^{(\gamma)}] = 0$$

We can now choose a new set of three orthogonal axes by going from (x, y, z) to (x', y', z) and we consider the new one-time decoherent family $\text{Fam}^{(x', y', z)}$ whose maximally fine-grained histories are associated to the projection operators $\{P_1^{\alpha'_1}\} = \{P_{\alpha'}, P_{\beta'}, P_{\gamma'}\}$, which are such that:

$$\Sigma_{x'}^2 = P_{\alpha'} + P_{\beta'}; \qquad \Sigma_{y'}^2 = P_{\alpha'} + P_{\gamma'}; \qquad \Sigma_{z'}^2 = P_{\beta'} + P_{\gamma'} \equiv \Sigma_z^2 \qquad (7.16)$$

In the above equation $\Sigma_{x'}^2, \Sigma_{y'}^2, \Sigma_z^2$ are, as usual, the square of the spin components along the indicated directions. Now, since $P_\beta + P_\gamma = \Sigma_z^2 = P_{\beta'} + P_{\gamma'}$, the two coarse-grained histories $\text{His}^{(\beta+\gamma)} \equiv \{P_\beta + P_\gamma, t_1\} \in \text{Fam}^{(x, y, z)}$ and $\text{His}^{(\beta'+\gamma')} \equiv \{P_{\beta'} + P_{\gamma'}, t_1\} \in \text{Fam}^{(x', y', z)}$ are, de facto, the same history. Due to the validity of assumption (c), we can state that:

$$h[\text{His}^{(\beta')}] \vee h[\text{His}^{(\gamma')}] = h[\text{His}^{(\beta'+\gamma')}] = h[\text{His}^{(\beta+\gamma)}]$$
$$= h[\text{His}^{(\beta)}] \vee h[\text{His}^{(\gamma)}] = 0 \qquad (7.17)$$

i.e., since one and only one of the three maximally fine-grained histories must be true:

$$h[\text{His}^{(\beta')}] = h[\text{His}^{(\gamma')}] = 0; \qquad h[\text{His}^{(\alpha')}] = 1 \qquad (7.18)$$

We have thus shown that:

$$h[\Sigma_{z'}^2] = h[\Sigma_{y'}^2] = 1; \qquad h[\Sigma_{z'}^2] = 0 \qquad (7.19)$$

The argument can obviously be repeated for any pair of orthogonal triples having an axis in common. But Kochen and Specker have proved that when the game is played for more than 117 appropriately chosen triples one gets a contradiction.

Taking the risk of being pedantic we want to stress once more that our proof never requires to take into account different histories belonging to incompatible decoherent families. The key ingredient of the proof is the assumption that the same history (in the above example $\text{His}^{(\beta+\gamma)} \equiv \text{His}^{(\beta'+\gamma')}$), even when considered as belonging to incompatible decoherent families ($\text{Fam}^{(x, y, z)}$ and $\text{Fam}^{(x', y', z)}$), has the same truth value.

8. WHAT ABOUT IGUSES WHICH CAN COMMUNICATE?

Gell-Mann and Hartle have also put forward[24] the hypothesis that different IGUSes belonging to different—and in general incompatible—"quasi classical domains" may communicate with each other:

> If different realms [i.e., quasi-classical domains] exhibit IGUSes, we may investigate certain relations between them. Probabilistic predictions concerning the relationships between IGUSes in two different realms may be made by using a decohering set of histories containing alternatives referring to IGUSes in one realm and also alternatives referring to IGUSes in the other realm, provided the decoherence of the hybrid set follows from the initial conditions and Hamiltonian. The problem of drawing inferences in one realm concerning IGUSes using a distinct realm is then not so very different from that involved in ordinary searching for extraterrestrial intelligence. The IGUSes making use of one realm could conceivably draw inferences about IGUSes in another by seeking or creating "measurement situations" in which an alternative of one realm is correlated almost perfectly with an alternative from the other.

Before proceeding a specification is necessary. In the above sentence reference is made to "different realms" and, to a "hybrid set" satisfying the decoherence conditions containing alternatives referring to two realms. In what follows we will interpret the above statements as referring to the situation in which consideration is given to incompatible families (different realms) such that some (but not all) histories belonging to different families (the hybrid set) can be accommodated in a decoherent family. The only other alternative, i.e., that the different realms and the hybrid set are members of a unique decoherent family would render the argument useless.

Having made clear this point we remark that in the above sentence the authors tacitly assume that the *alternatives referring to IGUSes in one realm and also alternatives referring to IGUSes in the other realm* must possess the same truth value, both if they belong to the first realm or to the second one, obviously given for granted that in each realm and in the hybrid set the decoherence condition be satisfied. Actually, only if this happens the two IGUSes can be sure to be able to communicate correctly between them. More generally we stress that the very way Gell-Mann and Hartle contemplate as the mean for the two IGUSes to exchange information coincides with the way used by us to relate the spin properties referring to two different triplets of orthogonal directions with one direction in common: the two triplets are associated to two different—and incompatible—families of decoherent histories having a common coarse-graining, the history referring to the square of the spin along the axis they have in common. In Gell-Mann and Hartle's example the same happens, and it is just due to the common coarse-graining that the two IGUSes can exchange information. Thus, in a certain sense, the argument of these authors represents a further proof of the reasonableness and naturalness of the assumptions we have made to prove the theorem of Subsection 7.1.

But if one takes into account our conclusion, one realizes that the communication between IGUSes could turn out to be rather problematic. In fact, suppose that two IGUSes, which we will call IGUS1 and IGUS2, belong to realm 1 and realm 2, respectively, and these realms have a common coarse graining which can be used for exchanging information between them. Now one can consider IGUS3, belonging to realm 3, having a coarse graining in common with IGUS2, so that also these two IGUSes can communicate between them. We can go on assuming the existence of further IGUSes, each in a different realm, having a coarse graining in common with the one preceding it. The realm of the last of these additional IGUSes, IGUSN is assumed to have a coarse graining in common with the realm of IGUS1. Now IGUS1 can exchange information with IGUS2 which in turn can transfer it to IGUS3 and so on up to the moment in which the information reaches IGUSN which transmits this information to IGUS1. The example with the spin components analyzed in Subsection 7.2 shows that the information that is transmitted to IGUS1 may contradict the properties which are part of the information already possessed by IGUS1.

We are aware that this example looks more like a science-fiction story than as a physical compelling argument, but we have considered appropriate to discuss it for its implications concerning those attempts which take the DH approach as the appropriate one for dealing with a framework in which all statements about physical processes, properties possessed by physical systems, and so on, are reduced to information exchange between IGUSes.

9. SUMMARY, CONCLUSIONS AND PERSPECTIVES

We close this long paper by summarizing the fundamental questions regarding the interpretation of the DH approach which have not yet received (in our opinion) a clear cut answer, by stating once more which assumption we consider necessary and by suggesting the line to follow to work out a satisfactory theory.

- The probabilities assigned to decoherent histories are *probabilities of what*, *exactly*? The two possible alternatives are:

1. Just as in the case of Classical Mechanics *the probabilities associated to decoherent histories refer to objective properties of individual physical systems*. This implies that they must have a truth value and, as discussed in great detail, this in turn can be formalized by considering an

appropriate homomorphism between the histories of a decoherent family and the set $\{0, 1\}$. This seems to us the only reasonable solution if one pretends that the DH approach represents an actual improvement of the standard interpretation allowing to take a realistic attitude with respect to physical processes.

2. *The DH probabilities* (*or at least some of them*) *do not refer to objective properties of physical systems.* If this is the case the proponents of the DH approach should make clear "what they are probabilities of" and should put forward precise criteria to identify those histories which have a truth value. They should also make clear why these histories have a privileged status. Griffiths' proposal to attach truth values *only* to histories having probability equal to one or zero (if this is what he means) is useless. In our opinion no reasonable solution to the fundamental problems of quantum mechanics can be reached following such a line.

• The "single family rule" applies only to our knowledge about the properties of physical systems and to the way we can use such information to make (probabilistic) logical inferences or does it hold also for the properties themselves?

1. If *the "single family" rule concerns only our knowledge* and not the actual properties of physical systems which have a truth value completely independent from our knowledge, then such truth values cannot change by changing the decoherent family. In our language, the homomorphism is family independent. In our opinion this is the only reasonable attitude about this point.

2. If *the rule applies also to the properties of physical systems*, and, consequently the truth value of a history depends on the decoherent family to which it is considered to belong (in the precise sense that the truth value can change with the family) then there follows that the properties of which the theory speaks are not objective but they depend in a crucial way from the framework one chooses to describe the system. Such a situation is, in our opinion, unacceptable.

As we have openly stated, we believe that the choices labeled by 1 must be made for both questions we have raised. The reasons for this choices have been analyzed in detail and we urge the readers to take into account the precise ontological implications of dropping any one of them. As repeatedly remarked, we believe that any such choice renders the DH view at least as problematic as the standard interpretation of Quantum Mechanics.

Now we can analyze the final and crucial question we have repeatedly mentioned:

• Do all decoherent histories have the same ontological status, and, in particular, do they speak of properties objectively possessed by individual physical systems? Two answers are possible:

1. Yes, *all decoherent histories make reference to objective properties and have definite truth values*. If this is the case it is totally obscure how decoherent histories referring to physically unacceptable properties for macroscopic systems (such as their being in a superposition of macroscopically different states) can have any sensible physical meaning. Moreover we stress that the theorem of Section 7 shows that this choice is incompatible with the choices we have made concerning both previous questions. Since we consider such choices as necessary to have a meaningful theory which represents a real improvement on the standard interpretation of the theory, we are led to consider the alternative answer to the above question.

2. *Not all decoherent histories have a physical meaning*, only those belonging to an appropriate subset are associated to objective properties of physical systems, and, consequently, must have a truth value. In such a case non-ambiguous criteria to identify the histories which have truth values must be put forward. With reference to this point we remark that:

(a) *Omnès*' criterion to distinguish sensible families from senseless ones (which looks quite promising) does not work.

(b) *Griffiths*' criterion (when correctly interpreted) to attach truth values only to histories having probability one or zero is senseless since no realistic decoherent history can have such a probability of occurrence.

(c) The *Gell-Mann and Hartle*'*s* suggestion to introduce a measure of classicity to characterize the physically significative histories is extremely interesting but, up to now, it has not found a satisfactory formulation.

Concluding, the just indicated line seems to us the only interesting one: one must drastically reduce, by precise criteria, the set of decoherent families which can be considered and which are physically significative. Obviously, if one takes such an attitude the real problem is to work out a consistent criterion to identify the acceptable families.

ACKNOWLEDGMENTS

We acknowledge useful remarks and suggestions by the referee.

REFERENCES

1. J. S. Bell, in *Proceedings of the Nobel Symposium 65: Possible Worlds in Arts and Sciences* (Stockholm, 1986).

2. J. S. Bell, in *Sixty-Two Years of Uncertainty*, A. Miller ed. (Plenum, New York, 1990).
3. S. Goldstein, *Physics Today* (March 1998), p. 42.
4. R. B. Griffiths, *J. Stat. Phys.* **36**:219 (1984).
5. R. B. Griffiths, *Am. J. Phys.* **55**:11 (1987).
6. R. B. Griffiths, *Found. Phys.* **23**:1601 (1993).
7. R. B. Griffiths, *Phys. Rev. A* **54**:2759 (1996).
8. R. B. Griffiths, *Phys. Rev. A* **57**:1604 (1998).
9. R. Omnès, *J. Stat. Phys.* **53**:893 (1988); **53**:933 (1988); **53**:957 (1988); **57**:357 (1989).
10. R. Omnès, *Rev. Mod. Phys.* **64**:339 (1992).
11. R. Omnès, *The Interpretation of Quantum Mechanics* (Princeton University Press, 1994).
12. M. Gell-Mann and J. B. Hartle, in *Proceedings of the 3rd International Symposium on the Foundations of Quantum Mechanics in the Light of New Technology*, S. Kobayashi *et al.*, eds. (Physical society of Japan, Tokio, 1990); in *Complexity*, *Entropy and the Physics of Information*, *SFI Studies in the Sciences of Complexity*, Vol VIII, W. Zurek, ed. (Addison Wesley, Reading, 1990).
13. M. Gell-Mann and J. B. Hartle, in *Proceedings of the 25th International Conference on High Energy Physics*, K. K. Phua *et al.*, eds. (World scientific, Singapore, 1991).
14. M. Gell-Mann and J. B. Hartle, *Phys. Rev. D* **47**:3345 (1993).
15. H. F. Dowker and A. Kent, *Phys. Rev. Lett.* **75**:3038 (1995); *J. Stat. Phys.* **82**:1575 (1996).
16. B. d'Espagnat, *Phys. Lett. A* **124**:204 (1987); *J. Stat. Phys.* **56**:747 (1989); *Found. Phys.* **20**:1147 (1990).
17. S. Goldstein and D. N. Page, *Phys. Rev. Lett.* **74**:3715 (1995).
18. R. P. Feynman, R. B. Leighton, and M. L. Sands, *The Feynman Lectures on Physics*, Vol. III, *Quantum Mechanics* (Addison-Wesley, 1963–1965).
19. Y. Aharonov and L. Vaidman, *J. Phys. A* **24**:2315 (1991).
20. R. Griffiths and J. B. Hartle, *Phys. Rev. Lett.* **81**:1981 (1998).
21. A. Kent, *Phys. Rev. Lett.* **78**:2874 (1997).
22. A. Kent, *Phys. Rev. Lett.* **81**:1982 (1998).
23. I. Giardina and A. Rimini, *Found. Phys.* **26**:973 (1996).
24. M. Gell-Mann and J. B. Hartle, preprint gr-qc/9404013, v.3, May 5, 1996.
25. S. Kochen and E. P. Specker, *J. Math and Mech.* **17**:59 (1967).
26. J. S. Bell, *Found. Phys.* **12**:989 (1982).
27. D. Dürr, S. Goldstein, and N. Zanghí, in *Bohmian Mechanics and Quantum Theory: An Appraisal*, J. T. Cushing, A. Fine, and S. Goldstein, eds. (Kluwer Academic Press, 1996); M. Daumer, D. Dürr, S. Goldstein, and N. Zanghí, *Erkenntnis* **45**:379 (1997).
28. A. Bassi and G. C. Ghirardi, *Phys. Lett. A* **257**:247 (1999).